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Escape from Geometric Lock-in in GCD-Augmented Fibonacci Recurrences

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Abstract

We study the recurrence $C(n) = C(n-1) + C(n-2) + \gcd\big(C(n-1), C(n-2)\big)$ through its states, the coprime pairs obtained by dividing consecutive terms by their gcd. A step is an amplification when this gcd grows, and once the states reach a cycle the sequence grows exactly geometrically. The author's earlier work left open whether every seed reaches a cycle. We show that it does not: the amplifications can stop, and this escape can be certified by a finite computation. Between amplifications the shifted states evolve by the Fibonacci matrix and their norm N alternates in sign, so a further amplification can only come from a prime dividing $N^2 - 1$, and finitely many orbit computations modulo these primes decide, once $N^2 - 1$ is factored, whether it ever comes. The seed (11, 5) is never amplified: $C(n) = 6L_{n-1} - 1$ for all n, with L_n the Lucas numbers. The seed (1, 135) joins this orbit after three amplifications, and infinitely many seeds escape. Every escaping orbit has N divisible by 30, and its growth ratio tends to the golden ratio. Among the 24,463 coprime seeds with entries at most 200, at least 956 escape (907 certified within 1,200 steps and 49 by an extended check), 77% of them on the orbit of (11, 5), while 2,320 remain undecided.

Keywords: GCD-augmented Fibonacci recurrence; geometric lock-in; escaping orbits; Lucas numbers; shifted Fibonacci sequences; greatest common divisor; Pisano period; Collatz-type maps

2020 Mathematics Subject Classification: 11B37, 11B39, 11A05, 11Y55

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1. Introduction

The recurrence $C(n) = C(n-1) + C(n-2) + \gcd\big(C(n-1), C(n-2)\big)$ adds to the Fibonacci rule the gcd of the two preceding terms. With seeds (1, 1) it is OEIS A083658 [1], which grows by exactly 3 every two steps. The author's earlier work [2–4] explained this by dividing consecutive terms by their gcd: the resulting coprime pairs, called states, follow a map R, and a

step is an amplification when the gcd grows. Once the states reach a cycle of R, the sequence grows by an exact factor G every q steps [4, Lemma 3(b)]; this is geometric lock-in. Lock-in replaces the golden ratio by rational growth factors such as 2, 3 and 7, those of the three cycles with one amplification per period [4, Theorem 10].

Whether every seed reaches a cycle was left open [4, Section 6]. In the census of [4, Table 2], 10.0% of the coprime seeds with entries at most 120, and 13.4% of those with entries at most 200, showed no repeated state within 1,200 steps, but no seed was proved to avoid a cycle. The anomalous seeds $(1, p)$ with p odd and divisible by 3 [2, Section 4.3] and a question on their transients [3, Section 6.5] belong to the same problem.

The question resembles divergence for Collatz-type maps, where it is a hard open problem: no divergent trajectory of the 3x + 1 map is known [5], and for the 5x + 1 map almost all trajectories are expected to diverge, yet none has been proved to, not even that of 7 [6]. For integer recurrences with conditional division, Greene [7] proved that every positive solution of a family with a fixed divisor is unbounded or periodic, and Kirschenhofer, Pethő and Thuswaldner [8] found explicit divergent seeds for a floor recurrence involving the golden ratio; Rowland's recurrence [9] and the EKG sequence [10] are other sequences defined through a gcd. Here $C(n)$ always grows, so the question is whether the states become periodic, and the divisor is a gcd read off the state.

We answer the question in the negative:

- Between amplifications the shifted state evolves by the Fibonacci matrix, and its norm N alternates in sign (Lemma 1).
- Only a prime dividing $N^2 - 1$ can amplify the next step (Lemma 2), a special case of the gcd bounds of Chen and Pan [11], so orbit computations modulo finitely many primes decide whether and when the next amplification occurs (Theorem 3).
- Escape forces $30 \mid N$ (Proposition 4), and an escaping sequence eventually grows like φ^n (Corollary 5).
- The seed (11, 5) is never amplified, so $C(n) = 6L_{n-1} - 1$ for all n; the seed (1, 135) joins this orbit after three amplifications (Theorem 6).
- Infinitely many seeds escape (Theorem 7), among them seven anomalous seeds below 400 (Corollary 8).
- A census splits the seeds without a detected cycle into certified escapes and undecided seeds (Section 5).

To the author's knowledge, no seed of this recurrence had previously been proved to escape; a search of the OEIS, arXiv and the journal literature on 21 October 2026 found none.

2. Preliminaries

Let a, b be positive integers and put $C(1) = a$, $C(2) = b$ and $C(n) = C(n-1) + C(n-2) + \gcd\big(C(n-1), C(n-2)\big)$ for $n \geq 3$; write $C_{a,b}$ for this sequence. Since $\gcd(kx, ky) = k\gcd(x, y)$, induction gives $C_{ka,kb} = k\, C_{a,b}$ [2, Theorem 4], so it suffices to consider the following.

A1. The seeds a, b are positive integers with $\gcd(a, b) = 1$.

We use the notation of [4, Section 2]. For $n \geq 2$ let $d_n = \gcd\big(C(n-1), C(n)\big)$; the *state* at index n is the coprime pair $\sigma_n = (\alpha_n, \beta_n) = (C(n-1)/d_n, C(n)/d_n)$, so $\sigma_2 = (a, b)$. For a coprime pair (α, β) the *step multiplier* is $g = \gcd(\beta, \alpha + 1)$, and
$R(\alpha, \beta) = \left(\frac{\beta}{g}, \frac{\alpha + \beta + 1}{g}\right)$.
By the cofactor factorization [3, Theorem 5 and Lemma 4], $\sigma_{n+1} = R(\sigma_n)$, $d_{n+1} = g(\sigma_n)\, d_n$ and $C(n) = d_n \beta_n$; in one line, $C(n+1) = d_n(\alpha_n + \beta_n + 1)$ and $\gcd(\beta_n, \alpha_n + \beta_n + 1) = \gcd(\beta_n, \alpha_n + 1)$. These are the only facts from [2] and [3] used in proofs; neither is among the items corrected in their versions of 21 October 2026, which we cite.

A step with $g > 1$ is an *amplification*. Steps are counted from 0: step j from a state σ goes from $R^j(\sigma)$ to $R^{j+1}(\sigma)$, so for a seed, step j goes from σ_{j+2} to σ_{j+3}. A cycle of R of period q has a *cycle multiplier* G, the product of its step multipliers. A sequence whose states enter such a cycle satisfies $C(n+q) = G\, C(n)$ for all large n; conversely, since σ_n is $\big(C(n-1), C(n)\big)$ divided by its gcd, this geometric lock-in gives $\sigma_{n+q} = \sigma_n$ [4, Lemma 3(b) and the remark after it].

F_n and L_n denote the Fibonacci and Lucas numbers ($F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$), $\varphi = \big(1 + \sqrt{5}\big)/2$, and Q is the matrix with rows $(0,1)$ and $(1,1)$, so that $Q(x, y) = (y, x + y)$.

For a state (α, β) we call $(u, v) = (\alpha + 1, \beta + 1)$ its *shifted state*, $N(u, v) = v^2 - uv - u^2$ its *norm* [4, Lemma 5], and $z = u\varphi + (v - u) \in \mathbb{Z}[\varphi]$ its *golden integer*. The field norm of $x + y\varphi$ is $x^2 + xy - y^2$, so the field norm of z is $N(u, v)$; it is never 0, because φ is irrational.

Definition 1. A seed *escapes* if its states have only finitely many amplifications. Every seed has exactly one of three *fates*: (a) its states enter a cycle of R; (b) it escapes; (c) it has infinitely many amplifications but never enters a cycle.

Fates (a) and (b) exclude each other because every cycle contains an amplification: β increases along non-amplifying steps [4, Section 6]. Escape is one of the two ways geometric lock-in can fail; the other is fate (c). Escape is not unboundedness, since $C(n)$ always increases.

Scope.

S1. Only the two-term recurrence is treated.
S2. Fate (c) is neither shown to occur nor excluded.
S3. Certificates are computed for explicit states; in the census, a seed can remain undecided when a norm is too large to factor or no certificate is reached within the step limit.

3. Amplifications and a Finite Certificate

Lemma 1. Let σ be a state with shifted state (u, v) and golden integer z, and suppose that steps $0, \ldots, k-1$ from σ are not amplifications. Then for $0 \leq j \leq k$ the state $R^j(\sigma)$ has shifted state $Q^j(u, v)$, golden integer $z\varphi^j$ and norm $(-1)^j N(u, v)$.

Proof. A non-amplifying step maps (α, β) to $(\beta, \alpha + \beta + 1)$, so the shifted state goes to $(v, u + v) = Q(u, v)$ [4, Lemma 1]. Since $\varphi^2 = \varphi + 1$, we get $z\varphi = v\varphi + u$, the golden integer of $(v, u + v)$. The norm changes sign because φ has norm -1. ■

Lemma 2. Let σ have shifted state (u, v) and norm N_0, and put $(u_j, v_j) = Q^j(u, v)$ for all $j \geq 0$.

(a) The first amplification from σ is at the least j with $\gcd(v_j - 1, u_j) > 1$, and that gcd is its step multiplier. If there is no such j, then σ is never amplified again.

(b) A prime r divides $\gcd(v_j - 1, u_j)$ if and only if $(u_j, v_j) \equiv (0,1) \pmod{r}$, that is, $z\varphi^j \equiv 1 \pmod{r}$. In that case $r \mid N_0 - (-1)^j$; in particular $r \mid N_0^2 - 1$.

(c) If $N_0 = \pm 1$, then one of steps 0,1,2 is an amplification.

Proof. The step multiplier of a state with shifted state (u', v') is $\gcd(\beta', \alpha' + 1) = \gcd(v' - 1, u')$; induction with Lemma 1 gives (a). In (b), the pair $(0,1)$ has norm 1, while the norm at step j is $(-1)^j N_0$ by Lemma 1. For (c), the positive solutions of $(v^2 - uv - u^2)^2 = 1$ are the consecutive Fibonacci pairs [12], so $(u, v) = (F_k, F_{k+1})$ with $k \geq 3$ and $(u_j, v_j) = (F_{k+j}, F_{k+j+1})$. Since F_m is even exactly when $3 \mid m$, the step $j \in \{0,1,2\}$ with $3 \mid k + j$ has u_j even and v_j odd, so 2 divides its gcd. ■

With $G_j = u_j$, the gcd in Lemma 2 is $\gcd(G_{j+1} - 1, G_{j+2} - 1)$, a gcd of consecutive terms of a shifted generalized Fibonacci sequence. That it divides $N_0 \mp 1$ is a special case of the bounds of Chen and Pan for shifted Horadam sequences [11, Example 6]. Part (c) is the analogue of the classical unbounded case of $\gcd(F_n + a, F_{n+1} + a)$ for $a = \pm 1$ [13–16]. What the dynamics needs, and Lemma 2 adds, is the exact location of the next amplification.

For a prime r, the Pisano period $\pi(r)$ is the least $k \geq 1$ with $Q^k \equiv I \pmod{r}$, equivalently the order of φ modulo r. It satisfies $\pi(5) = 20$; $\pi(r) \mid r - 1$ if $r \equiv \pm 1 \pmod 5$; and $\pi(r) \mid 2(r + 1)$ if $r \equiv \pm 2 \pmod 5$ [17].

Theorem 3 (certificate). Let σ have shifted state (u, v) with $N_0^2 \neq 1$. For each prime $r \mid N_0^2 - 1$ let $J_r = \{ j \geq 0 : Q^j(u, v) \equiv (0,1) \pmod{r} \}$, and let J be the union of the sets J_r. Each J_r is empty or an arithmetic progression with difference $\pi(r)$, and it is determined by computing $Q^j(u, v) \bmod r$ for $0 \leq j < \pi(r)$. If J is nonempty, the next amplification from σ is at step $\min J$. If J is empty, σ is never amplified again, and the seed escapes.

Proof. By Lemma 2, an index j is amplifying for the sequence $Q^j(u, v)$ exactly when $j \in J$, and by Lemma 2(a) the first such index is the next amplification. Since Q^k has rows (F_{k-1}, F_k) and (F_k, F_{k+1}), we have $Q^k(0,1) \equiv (0,1) \pmod{r}$ exactly when $Q^k \equiv I$, that is, when $\pi(r) \mid k$; so $J_r = \{\min J_r + m\,\pi(r) : m \geq 0\}$. ■

The test asks whether z lies in the subgroup generated by φ in $(\mathbb{Z}[\varphi]/r)^\times$; for large r, element orders or a discrete logarithm decide this faster than iteration. Large primes can produce late

amplifications: among the 54,795 coprime states with entries at most 300, ten first amplify only after step 3,000, the latest at step 47,359 (the state (275, 227), prime 87,119). No finite step limit alone can therefore prove escape. For the cycles (3, 5) and (29, 47) of [4, Theorem 10], Theorem 3 predicts the amplifying step $q-1$ (for (29, 47), $N_0 = -36$ and $7 \mid N_0 + 1$); the fixed point (1, 2) has norm -1 and is covered by Lemma 2(c).

Proposition 4 (obstruction primes). Call a prime r *full* if the vectors of norm ± 1 modulo r are exactly the residues $(F_j, F_{j+1}) \bmod r$.

(a) The primes 2, 3 and 5 are full. A prime $r \equiv \pm 1 \pmod 5$ is never full, and an odd prime $r \equiv \pm 2 \pmod 5$ is full if and only if $\pi(r) = 2(r+1)$.

(b) If r is full and $N_0 \equiv \pm 1 \pmod r$, an amplification occurs within $\pi(r)$ steps.

(c) Consequently, after the last amplification of an escaping seed, $30 \mid N_0$ and $N_0 \not\equiv \pm 1$ modulo every full prime.

Proof. (b) We have $(u, v) \equiv Q^m(0,1)$ for some $0 \le m < \pi(r)$, so $Q^{\pi(r)-m}(u, v) \equiv (0,1)$, and Lemma 2 applies.

(a) By Cassini's identity, $N(F_j, F_{j+1}) = F_{j-1}F_{j+1} - F_j^2 = (-1)^j$, so the $\pi(r)$ residues (F_j, F_{j+1}) all have norm ± 1. For odd $r \neq 5$, $\mathbb{Z}[\varphi]/r$ is $\mathbb{F}_r \times \mathbb{F}_r$ when $r \equiv \pm 1 \pmod 5$, with $r-1$ elements of each nonzero norm, and $\mathbb{F}_{r^2}$ otherwise, with $r+1$. So there are $2(r-1)$ or $2(r+1)$ vectors of norm ± 1. In the first case $\pi(r) \le r-1$. For $r = 2,3,5$ there are 3, 8 and 20 such vectors and $\pi(r) = 3,8,20$.

(c) Every integer is $\equiv 0$ or ± 1 modulo 2 and modulo 3, and $N_0 \equiv (v+2u)^2 \pmod 5$ is $\equiv 0$ or ± 1. Apply (b) to an infinite stretch. ■

The full primes below 100 are 2, 3, 5, 7, 13, 17, 23, 37, 43, 53, 67, 73, 83 and 97; there are 489 below 10^4. Equivalently, an odd prime $r \equiv \pm 2 \pmod 5$ is full when φ generates the subgroup of norm ± 1 in $\mathbb{F}_{r^2}^{\times}$, which has order $2(r+1)$. Full primes are thus the inert analogue of the primes with a Fibonacci primitive root, a primitive root g with $g^2 \equiv g+1$ [18]; whether infinitely many primes are full is a variant of Artin's primitive root conjecture for quadratic units [19, Section 6.1.1]. Proposition 4 also explains why amplifications are frequent: from any state whose norm is prime to 30, the primes 2, 3 and 5 force an amplification within 20 steps.

Corollary 5 (fates).

(a) A seed of fate (a), with cycle period q and cycle multiplier G, satisfies $C(n+q) = G\,C(n)$ for all large n.

(b) Let a seed escape, and let σ_{n_0} be its state after the last amplification, with shifted state (u, v) and $d = d_{n_0}$. Then $30 \mid N(u, v)$, the states σ_n for $n \ge n_0$ are distinct, and for all $j \ge 0$ $C(n_0 + j) = d\,(uF_j + vF_{j+1} - 1)$,

so $C(n+1)/C(n) \to \varphi$.

(c) Iterate the states. At the seed and after each amplification, apply Theorem 3 whenever $N_0^2 \neq 1$; stop at a repeated state or when $J = \emptyset$. This recognizes the seeds of fates (a) and (b) after finitely many operations, and it runs forever on fate (c).

Proof. (a) is [4, Lemma 3(b)].

(b) Since d is constant from n_0 on, Lemma 1 gives the shifted states $Q^j(u,v) = \left(uF_{j-1} + vF_j,\ uF_j + vF_{j+1}\right)$, and $C(n) = d_n\beta_n$. So C/d satisfies the Leonardo-type recurrence $x_{j+1} = x_j + x_{j-1} + 1$ [3, Proposition 3], [20]. The condition $30 \mid N(u,v)$ is Proposition 4(c).

(c) For an escaping seed, the stretch from the state after its last amplification (or from the seed, if it is never amplified) is infinite. So $N_0^2 \neq 1$ by Lemma 2(c), and Theorem 3 returns $J = \emptyset$ there. ■

4. Escaping Seeds

Theorem 6.

(a) The seed (11, 5) is never amplified, and $C(n) = 6L_{n-1} - 1$ for all $n \geq 1$. Equivalently, $\gcd(6L_n - 1, 6L_{n+1} - 1) = 1$ for all $n \geq 0$.

(b) The seed (1, 135) has exactly three amplifications, at steps 2, 8 and 9, each with step multiplier 3. After them, $C(n) = 27\,(6L_{n-3} - 1)$ for all $n \geq 11$.

In both cases the states never repeat and $C(n+1)/C(n) \to \varphi$.

Proof.
(a)

1. The seed (11, 5) has shifted state $(12,6) = 6(L_0, L_1)$, golden integer $z = 12\varphi - 6 = 6\sqrt{5}$ and norm $N_0 = -180$. Both $N_0 - 1 = -181$ and $N_0 + 1 = -179$ are prime, so Theorem 3 needs only $r = 179$ and $r = 181$.
2. Both primes are $\equiv \pm 1 \pmod 5$, so $x^2 - x - 1$ has a root ρ modulo r; we take $\rho \equiv 105$ for $r = 179$ and $\rho \equiv 14$ for $r = 181$. The map $\varphi \mapsto \rho$ is a ring homomorphism $\mathbb{Z}[\varphi] \to \mathbb{F}_r$ sending $\sqrt{5} = 2\varphi - 1$ to $2\rho - 1$ and z to $z_1 = 6(2\rho - 1)$. A step $j \in J_r$ requires $z_1\rho^j \equiv 1$.
3. $r = 179$. By Lemma 2(b) only odd j can lie in J_{179}, because $179 \mid N_0 + 1$. Here $2\rho - 1 \equiv 30$ and $z_1 \equiv 180 \equiv 1$, so $j \in J_{179}$ would need $\rho^j \equiv 1$. But ρ is a primitive root modulo 179, which forces $178 \mid j$, so j is even. Hence $J_{179} = \emptyset$.
4. $r = 181$. Here $2\rho - 1 \equiv 27$ and $z_1 \equiv 162$ with $z_1^2 \equiv 180 \equiv -1$, so z_1 has order 4. But ρ has order 45, which is odd, so no power of ρ equals z_1^{-1}. Hence $J_{181} = \emptyset$.
5. By Theorem 3, (11, 5) is never amplified. Its shifted states are $Q^k(12,6) = 6(L_k, L_{k+1})$, so $\beta = 6L_{k+1} - 1$ with $d = 1$, which gives (a).

(b)

1. The terms $C(1), \ldots, C(14)$ are 1, 135, 137, 273, 411, 687, 1101, 1791, 2895, 4689, 7587, 12285, 19899 and 32211.
2. The states $\sigma_4 = (137,273)$, $\sigma_{10} = (965,1563)$ and $\sigma_{11} = (521,843)$ have step multiplier 3; every other step up to σ_{12} has step multiplier 1. This gives $\sigma_{12} = (281,455)$ with $d_{12} = 27$.
3. Since $(282,456) = 6(L_8, L_9)$, the state σ_{12} equals $R^8(11,5)$, which by (a) is never amplified.
4. Hence $C(n) = 27\beta_n = 27(6L_{n-3} - 1)$ for $n \geq 12$, and also $C(11) = d_{12}\alpha_{12} = 27 \cdot 281 = 27(6L_8 - 1)$. The growth statement is Corollary 5(b). ■

Theorem 7. Infinitely many coprime seeds escape. For every $t \geq 1$, the seed $(48894t - 1, 78961t)$ is amplified once, with step multiplier $281t$, and is never amplified again.

Proof. Let (α', β') be a state with $\beta' > \alpha'$, and let $g \geq 1$ satisfy $g(\beta' - \alpha') \geq 2$ and $\gcd(g(\beta' - \alpha') - 1, \alpha') = 1$. Put $\alpha = g(\beta' - \alpha') - 1$ and $\beta = g\alpha'$.

- Then $\gcd(\beta, \alpha + 1) = g\gcd(\alpha', \beta' - \alpha') = g$.
- Also $\gcd(\alpha, \beta) = 1$, and $R(\alpha, \beta) = (\alpha', \beta')$.

Take $(\alpha', \beta') = (281,455)$ and $g = 281t$. Then $\alpha' \mid g$, so the gcd condition holds, and $(\alpha, \beta) = (281 \cdot 174\, t - 1,\ 281^2 t)$. The state (281, 455) is never amplified, by Theorem 6. ■

The same construction turns every certified escape into a sparse infinite family of escaping seeds; their density is the subject of Conjecture 9.

For seeds $(1, p)$, the closed forms of [2] cover even p and odd p prime to 3. The remaining class, odd p divisible by 3, was left open [2, Section 4.3]. Its members can reach the 2-cycle (3, 5), the 4-cycle (29, 47) or a cycle of period 22 [4, Proposition 12 and Section 5].

Corollary 8.

(a) For $p = 135,159,195,219,321,375$ and 387, the seed $(1, p)$ escapes (Table 1).

(b) Whether such a seed locks in is not determined by $v_3(p)$, the exponent of 3 in p. The seeds $p = 3$ and 21 have $v_3(p) = 1$ and reach (3, 5), while $p = 159,195,219,321$ and 375 also have $v_3(p) = 1$ and escape. So the transient length cannot be bounded in terms of $v_3(p)$, which answers the question in [3, Section 6.5] in the negative.

Proof. Table 1 gives, for each p, the state after the last amplification and the primes dividing $N_0^2 - 1$. Theorem 3 with these primes returns $J = \emptyset$. For $p = 3$ and 21 the states reach (3, 5) after a few steps. ■

Table 1. The seeds $(1, p)$ with $p < 400$, p odd and divisible by 3, certified to escape within 1,200 steps (the others reach a cycle or remain undecided, for example $p = 45$, Figure 1): number of amplifications (Ampl.), index n_0 and state σ_{n_0} after the last one, its norm N_0, and the primes $r \mid N_0^2 - 1$ used by Theorem 3.

p	$v_3(p)$	Ampl.	n_0	σ_{n_0}	N_0	Primes r
135	3	3	12	(281, 455)	−180	179, 181
159	1	45	172	(1393991, 2255525)	−1,359,180	1,359,179, 1,359,181
195	1	34	129	(25115, 40637)	−30,420	19, 29, 1,049, 1,601
219	1	3	9	(107, 173)	−180	179, 181
321	1	10	38	(281, 455)	−180	179, 181
375	1	11	54	(281, 455)	−180	179, 181
387	2	7	36	(281, 455)	−180	179, 181

Five of the seven seeds join the orbit of Theorem 6(a): $(107,173) = (6L_6 - 1, 6L_7 - 1)$ and $(281,455) = (6L_8 - 1, 6L_9 - 1)$. For $p = 159$ the certificate uses the twin primes 1,359,179 and 1,359,181, as for $p = 135$ it uses 179 and 181. Heuristically, such twin primes recur because they leave only two primes that can end the stretch.

5. Census

We took the 8,771 coprime seeds with $1 \le a, b \le 120$ (box 120) and the 24,463 with $1 \le a, b \le 200$ (box 200), as in [4, Table 2], and iterated R for at most 1,200 steps. Each seed was classified as a cycle (a state repeats), a certified escape (Theorem 3 returns $J = \emptyset$, at the seed or after an amplification) or undecided (neither).

Certificates were attempted when $|N_0| + 1$ has at most 30 digits, with orbit tests by iteration for $r \le 3 \cdot 10^6$ and by a discrete logarithm for $r \equiv \pm 1 \pmod 5$ up to 10^{12}. A larger prime leaves the seed undecided, so the counts of certified escapes are lower bounds. The cycle rows of Table 2 reproduce [4, Table 2] exactly, and its last two rows split the 873 and 3,276 seeds with no repeated state. No count changes when the step limit is doubled to 2,400, and every certified escape has $30 \mid N_0$, as Proposition 4 requires.

Table 2. Fates of the coprime seeds in box 120 and box 200 after at most 1,200 steps of R, under the certificate limits above.

Fate	Box 120: seeds	Share	Box 200: seeds	Share
Fixed point (1, 2)	912	10.4%	1,810	7.4%
2-cycle (3, 5)	3,226	36.8%	7,918	32.4%
4-cycle (29, 47)	3,153	35.9%	9,386	38.4%
Period-22 cycle through (97, 157)	607	6.9%	2,071	8.5%
Period-65 cycle through (4003, 6477)	0	0.0%	2	<0.1%
Certified escape	235	2.7%	907	3.7%
Undecided	638	7.3%	2,369	9.7%
Total	8,771	100%	24,463	100%

As an extended check, each undecided seed was re-examined at its state after the last amplification within 2,400 steps: $N_0 \mp 1$ was factored with PARI/GP for norms of up to 100 digits, with 20 seconds per factorization, and the orbit tests used element orders, for primes of any size. In box 120 and box 200, 14 and 49 more seeds are certified to escape, giving 249 (2.8%) and 956 (3.9%); 612 and 2,222 are proved to amplify again after the first 2,400 steps; 12 and 98 remain unresolved. So box 200 has at least 956 escaping and 2,320 undecided seeds.

Computational details. All computations were rerun with Python 3.13.17, SymPy 1.14.0 and PARI/GP 2.17.2 [21] (cypari2 2.2.4) and reproduce every number in this paper. Census certificates factor with SymPy and test only primes below 10^{12}. Every verdict of the extended check was recomputed with PARI/GP's flag factor_proven set, which proves every prime factor, including those behind element orders; none changed. Every prime in every certificate of this paper was also proved prime with PARI/GP's isprime; these two checks are not part of the archived script [22]. A seed is proved to amplify again when Theorem 3, applied after its last amplification within 2,400 steps, returns $J \neq \emptyset$.

Quiet stretches are weak evidence of escape: of the 758 undecided seeds of box 200 with no amplification in their second 1,200 steps, 672 are proved to amplify again later. The undecided seeds keep amplifying, a median of 381 times in 2,400 steps, and their cofactors β then have 39 to 504 digits (median 211). They are candidates for fate (c) or for very long transients (Figure 1).

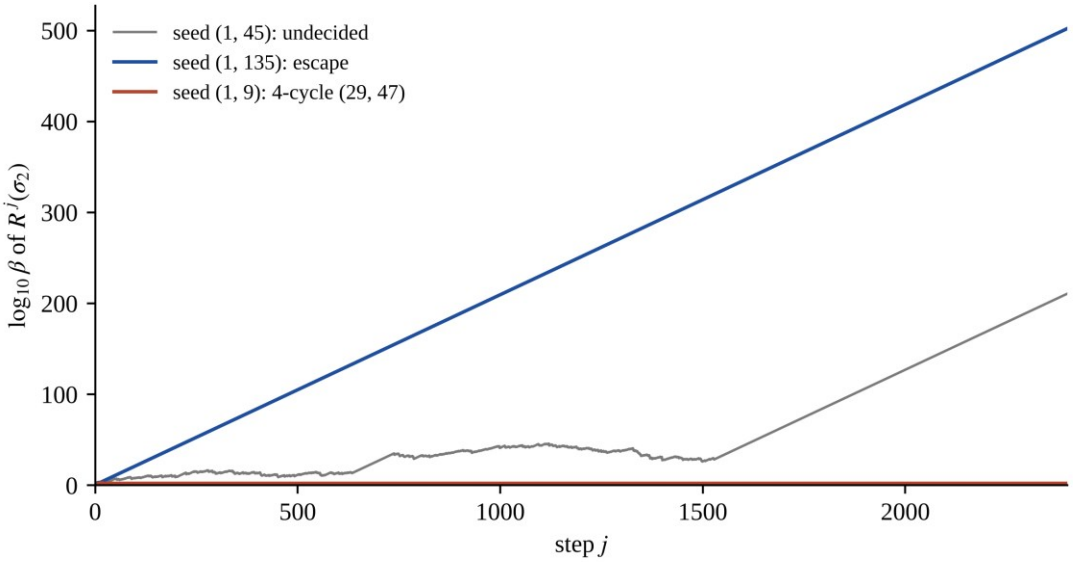

Figure 1. The cofactor β of $R^j(\sigma_2)$ against the step j, on a logarithmic scale. The seed (1, 9) reaches the 4-cycle (29, 47), so β stays bounded. The seed (1, 135) is last amplified at step 9, after which $\log_{10}\beta$ grows with slope $\log_{10}\varphi \approx 0.209$. The seed (1, 45) is undecided: not amplified since step 1,527, it will be amplified again by Theorem 3.

Escaping orbits are few. Reducing each certified state by $z \mapsto z/\varphi$ to a canonical member of its orbit, the 956 certified escapes of box 200 lie on 20 orbits, and the orbit of Theorem 6(a) carries 739 of them (77%). For the seeds $(1, p)$ with $p \leq 2{,}000$, under the census limits and 2,400 steps, 43 are certified to escape (4, 9, 22, 31 and 43 with $p \leq 250{,}500{,}1{,}000{,}1{,}500{,}2{,}000$), again a

lower bound. All are odd multiples of 3, as the closed forms of [2] require, and 31 of them lie on the orbit of Theorem 6(a).

The orbit of Theorem 6(a) is one of a family. For $m \geq 2$, the seed $(2m-1, m-1)$ has golden integer $m\sqrt{5}$ and norm $-5m^2$. By Proposition 4 it can avoid amplification forever only if $6 \mid m$. For $m \leq 3{,}000$, Theorem 3, with no limit on the primes, decides every m: the seed is never amplified for exactly 53 values, the first being 6, 18, 42, 78 and 102, and for them $C(n) = mL_{n-1} - 1$ for all n. The counts up to 500, 1,000, 2,000 and 3,000 are 11, 21, 44 and 53.

Conjecture 9. (i) There are infinitely many m with $\gcd(mL_n - 1, mL_{n+1} - 1) = 1$ for all $n \geq 0$. (ii) There are infinitely many p for which the seed $(1, p)$ escapes. (iii) The escaping seeds have positive lower density among the coprime seeds with $1 \leq a, b \leq B$ as $B \to \infty$.

The evidence is the counts above and the share of certified escapes in box B at 1,200 steps, which grows with B: 1.6%, 2.3%, 2.7%, 3.1% and 3.7% for $B = 50{,}100{,}120{,}150$ and 200. These shares are lower bounds, and they cover $B \leq 200$ only.

6. Discussion

The mechanism is linear: between amplifications z is multiplied by φ, so whether a stretch ever ends is a question about z modulo the finitely many primes dividing $N_0^2 - 1$, each periodic with period $\pi(r) \leq 4r$. For the 5x + 1 map no comparable certificate is known [6]; there the parities along a trajectory depend on ever higher binary digits of the starting value.

The limitations follow the scope labels.

(S1) For the Tribonacci analogue of [4, Section 6], stretches are linear in a 3×3 matrix, but the norm form becomes cubic, and the analogue of Lemma 2 is left for future work. More generally, any recurrence that is linear between amplifications, with amplifications triggered by congruences modulo primes dividing an integer fixed along the stretch (here $N_0^2 - 1$), admits a certificate of the same kind once that integer is factored.

(S2) Fate (c) remains open. Proving it for one seed, the analogue of proving that a single Collatz trajectory diverges, would need a property of the states that persists from one stretch to the next, forces every stretch to end and rules out a return to an earlier state; Theorem 3 decides only one stretch at a time. Deciding fate (c) for a single seed is the main open problem.

(S3) A certificate requires the prime factors of $N_0 \mp 1$, proved prime (Section 5). Seeds remain undecided when these are out of reach, so the census gives lower bounds; larger factoring caps would decide more of them. The evidence for Conjecture 9 comes from boxes with $B \leq 200$, the seeds $(1, p)$ with $p \leq 2{,}000$ and $m \leq 3{,}000$ only.

By the argument of Theorem 6, whether the orbit of $m\sqrt{5}$ escapes (Conjecture 9(i)) depends only on multiplicative orders modulo the primes dividing $25m^4 - 1$; a density heuristic for these orders, of the Artin type surveyed in [19], is the natural next step. The census labels every seed in box 200 as cycling, certified to escape, or undecided, so it is also a labelled dataset; whether

learned predictors, such as the in-context learners analysed in [23–26], can predict the fate of a seed without a certificate is left open.

7. Conclusions

Not every seed of the GCD-augmented Fibonacci recurrence reaches geometric lock-in. A finite certificate, built on the Fibonacci norm form, shows that the seed (11, 5) is never amplified, so $C(n) = 6L_{n-1} - 1$ for all n, that the seed (1, 135) and six further anomalous seeds below 400 escape, and that infinitely many seeds escape. Every escaping orbit has norm divisible by 30, and escaping sequences grow like φ^n. In box 200, at least 956 of the 24,463 coprime seeds escape: 907 are certified within 1,200 steps (Table 2) and 49 more by the extended check. Whether the 2,320 undecided seeds, which keep amplifying, ever reach a cycle remains open.

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Every numerical result, both tables and Figure 1 can be reproduced with the Python script reproduce.py, archived on Zenodo with its GitHub source [22].

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